Weakly bound states of neutrons in gravitational fields

Avas V. Khugaev[a†]

Bogoliubov Laboratory of Theoretical Physics,

Joint Institute of Nuclear Research, 141980 Dubna, Russia

Renat A. Sultanov[‡]

Department of Information Systems and BCRL,

St. Cloud State University, 367B Centennial Hall,

720-4th Avenue South, St. Cloud, MN 56301-4498, USA

Dennis Guster[§]

Department of Information Systems and BCRL,

St. Cloud State University, 367C Centennial Hall,

720-4th Avenue South, St. Cloud, MN 56301-4498, USA

(Dated: December 30, 2010)

Abstract

In this paper a quantum-mechanical behaviour of neutrons in gravitational fields is considered. A first estimation is made using the semiclassical approximation, neglecting General Relativity, magnetic and rotation effects, for neutrons in weakly bound states in the weak gravitational field of the Earth. This result was generalized for a case, in which the Randall - Sundrum correction to Newton's gravitational law on the small scales was applied. Application of the results to Neutron Star physics is considered and further possible perspectives are discussed.

PACS numbers: 03.65.Ta, 98.80.Cq

[a] Permanent address: Institute of Nuclear Physics, 100214 Tashkent, Uzbekistan

[†] avaskhugaev@yahoo.com

[‡] rasultanov@stcloudstate.edu

[§] dcguster@stcloudstate.edu

I. INTRODUCTION

The idea concerning the potential of neutron storage by using of UCN (ultra cold neutrons), probably was first suggested in 1959 by Ya.B. Zeldovich [1]. In his work a simple estimate concerning the possibility of UCN conservation in the container was devised. It was shown that a UCN (with wave length larger than 500 $\dot{A}$ and effective temperature less than 10^{-3} K) should be totally reflected from the reservoir walls (which were made from either carbon or beryllium material) when the neutrons are at velocities around $5m/s$ and less.

More rigorous results, obtained later, completely confirmed these estimations. In general, these results can be obtained in the framework of pure quantum - mechanical or optical approaches. In the quantum mechanical approach the potential of the reflecting surface is constructed on the basis of the average procedure of the pointlike Fermi quasipotential, given in the form:

$$u(r) = \frac{2\pi\hbar}{m} \cdot b\delta(\vec{r} - \vec{r}_i)$$

where b - is a neutron wave length of coherent scattering on the nuclei of the considered surface. In this approach, by strictly following the framework of quantum mechanics it was shown that the depth of neutron penetration in the surface material are significantly less, than the neutron wave length [2–4] and the neutron scattering on the surface can be considered as elastic. An alternative approach [5], was based on the optical analogy of light scattering on metals, such that all inelastic processes in the reflection can be effectively described by the imaginary part of the refraction coefficient. In this case the work of [1] serves a necessary theoretical premise for the starting point of all further experimental works based on the UCN technique. One of the first experimental results was obtained in 1968 by the JINR (Joint Institute of Nuclear Research, Dubna)[6]. The importance of this type of work is extremely high, because it opens a new and unique possibility to carry out highly precise experiments, such as accurate measurement of neutron lifetime or its electrical dipole moment [7] and etc.

The first proposal using experimental measurements with UCN in the gravitation field of the Earth was done by [8]. Therefore, there appears to be sound knowledge base from which to proceed given the achievements to date in both the theoretical and experimental fields. Certainly, the many recent works devoted to this research subject indicate that the solution

of the UCN storage problem can give us an unique capability to carry out a very precise experiments in the field of neutron physics and neutron interferometry in gravitational fields [9, 10]. In this paper, we will begin by reviewing some pertinent recent experimental results and measurements of UCN weakly bound quantum states in the weak gravitational field of the Earth [11–15]. After that, we will present a simple sketch of some of the other possible applications of these results to astrophysics and gravity. A more detailed description of some of other applications can be found in [16].

II. ENERGY OF WEAKLY BOUND STATES OF NEUTRONS IN THE EARTH'S GRAVITATIONAL FIELD.

Theoretical consideration of a neutron's energy spectrum and its wave functions in the Earth's gravitational field, as cited in the [15], can be found in a number of works, including, for example [17–19]. In this section, by using pure methodic reasoning, we present a simple and transparent derivation of the main relationships of these results. A description of weakly bound states in the gravitational field of the Earth follows in the Schrödinger Equation (SE) solution using a simple spherically symmetric case with gravitational potential energy, defined as:

$$\delta V(r) = \begin{cases} -\gamma M m\left(\frac{1}{r} - \frac{1}{R_0}\right), & r > R_0 \\ \\ \infty, & 0 \leq r \leq R_0 \end{cases} \tag{1}$$

where M and m are the Earth and neutron masses at rest respectively and r - is a distance from the Earth in cm up to the point where the neutron mass m is placed. In this case SE can be written in the following way:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\psi}{dr}\right) + \frac{2m}{\hbar^2}\Big[E - \delta V(r)\Big]\psi = 0 \tag{2}$$

where ψ is a neutron wave function and E is its corresponding energy. Substituting, in the normal way, $\psi = \frac{f(r)}{r}$ into the equation (2) we have:

$$\frac{d^2 f}{dr^2} + \frac{2m}{\hbar^2}\Big[E - \delta V(r)\Big] f = 0. \tag{3}$$

Using a simple approximation for the gravitational potential energy in the vicinity of the Earth surface, we can write:

$$\delta V(r) \rightarrow \delta V(x) = \gamma \frac{Mm}{R_0}\left[\frac{x}{R_0} + o\left(\frac{x}{R_0}\right)^2\right] \tag{4}$$

where $r = R_0 + x$, $\frac{x}{R_0} \ll 1$ and R_0 is an average radius of the Earth. In this approximation SE can be written as:

$$\frac{d^2 f}{dx^2} + (\alpha - \beta x) f = 0, \tag{5}$$

where we introduced α and β as parameters, defined by:

$$\alpha = \frac{2m}{\hbar^2} E; \quad \beta = 2\left(\frac{m}{\hbar}\right)^2 g; \quad g = \gamma \frac{M}{R_0^2} \tag{6}$$

By substituting a new variable $z = \alpha - \beta x$, we can finally write SE in the convenient form for the analytic solution:

$$\frac{d^2 f}{dz^2} - \frac{z}{(i\beta)^2} f = 0 \tag{7}$$

For this ordinary differential equation of the second order, as suggested by the theory of Bessel functions [20], there exists a regular solution which is presented in the following form:

$$f(z) = c_N \sqrt{z}\left[J_{\frac{1}{3}}\left(\frac{2z^{\frac{3}{2}}}{3\beta}\right) + J_{-\frac{1}{3}}\left(\frac{2z^{\frac{3}{2}}}{3\beta}\right)\right] \tag{8}$$

where c_N is a normalization coefficient of the wave function. From this last expression we can extract the energy spectrum of the neutrons weakly bound states by using a boundary condition at the point $x = 0$, which simply leads to the equation:

$$J_{\frac{1}{3}}\left(\frac{2\alpha^{\frac{3}{2}}}{3\beta}\right) + J_{-\frac{1}{3}}\left(\frac{2\alpha^{\frac{3}{2}}}{3\beta}\right) = 0 \tag{9}$$

If you assume that $\zeta_n \rightarrow \zeta = \frac{2\alpha^{\frac{3}{2}}}{3\beta}$ is such a number, then it satisfies the equation:

$$J_{\frac{1}{3}}\left(\zeta_n\right) + J_{-\frac{1}{3}}\left(\zeta_n\right) = 0 \tag{10}$$

which leads us to:

$$E_n = \left(\frac{9m}{8}(\hbar g \zeta_n)^2\right)^{\frac{1}{3}} = c_0(\zeta_n)^{\frac{2}{3}} \tag{11}$$

TABLE I. Comparison of our results with theoretical and experimental results, obtained in [13]. Energy values are given in peV units and the ζ_n value is dimensionless and E_n-th is calculated using expression (17)

n	ζ_n	E_n-th	E_n,[13] -exp.	E_n, our
1	2.3834	1.3767	1.4	1.4054
2	5.5105	2.42191	2.5	2.4573
3	8.6474	3.27356	3.3	3.3180
4	11.7868	4.0255	4.1	4.0794
5	14.9272	4.71261	–	4.7751

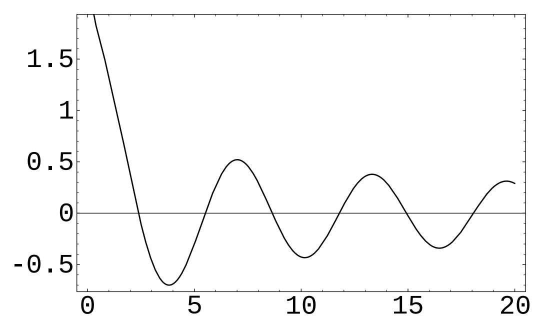

FIG. 1. ζ_n roots distribution for the $J_{\frac{1}{3}}(\zeta_n) + J_{-\frac{1}{3}}(\zeta_n) = 0$ equation.

where we introduced $c_0 = \left(\frac{9m}{8}(\hbar g)^2\right)^{\frac{1}{3}}$. By now using approximate values for the above defined constants, such as: $g \approx 9.80655$ ms^{-2}; $R_0 \approx 6.371 \cdot 10^6$ m; $mc^2 \approx 939.565330$ MeV and $\hbar c \approx 197.327053$ MeV$\cdot$ fm, we can calculate the first bound states for the E_n values and compare them with other results, including experimental measurements. These results are presented in Table 1, where the c_0 value is set to $c_0 \approx 0.757325 \cdot 10^{-12}$ eV. 10mm In Fig.1 an approximate distribution of the ζ_n roots is presented.

III. ASYMPTOTIC PROPERTIES

Given the solution obtained above it appears that it would be interesting to analyze its asymptotic properties. To do this we are able to use the properities of Bessel functions [20]:

$$J_\nu(y) = \left(\frac{2}{\pi y}\right)^{\frac{1}{2}} \cos\left[y - \frac{\pi}{2}(\nu + \frac{1}{2})\right] \tag{12}$$

by introducing $y \to \frac{2z^{\frac{3}{2}}}{3\beta}$ we can obtain:

$$f(y) \to 3^{\frac{5}{6}}\sqrt{\pi}\beta^{\frac{1}{3}}y^{-\frac{1}{6}} \sin\left[y + \frac{\pi}{4}\right] \tag{13}$$

from which we arrive at:

$$f(z) = c_N \cdot 2^{\frac{1}{3}}\left(\frac{\pi\beta}{2}\right)^{\frac{1}{2}} \cdot z^{-\frac{1}{4}} \sin\left[\frac{2\alpha^{\frac{3}{2}}}{3\beta} + \frac{\pi}{4}\right] \tag{14}$$

For the zero roots of this function we can immediately obtain from the last expression that:

$$\frac{2\alpha^{\frac{3}{2}}}{3\beta} + \frac{\pi}{4} = n\pi \tag{15}$$

and finally, using our notations for variable z we are led to:

$$E_n = mgx + \left(\frac{9m}{8}(\pi\hbar g)^2\right)^{\frac{1}{3}}\left(n - \frac{1}{4}\right)^{\frac{2}{3}} \tag{16}$$

Naturally, one might find the last expression to be quite interesting. This is true because the 1-st term of the expression can be precisely viewed as the classical part of an energy particle in a homogeneous gravitational field while the second term is a quantum mechanical contribution. In the case where $\hbar \to 0$ we have arrived at a pure classical result and in the case where $\hbar \neq 0$, but $x = 0$ we have exactly obtained results matching the energy spectrum from [13]:

$$E_n = \left(\frac{9m}{8}(\pi\hbar g)^2\right)^{\frac{1}{3}}\left(n - \frac{1}{4}\right)^{\frac{2}{3}} \tag{17}$$

IV. THE SEARCH FOR HIGH DIMENSION CONTRIBUTIONS.

As it was pointed out in [13], the possibility of making observations concerning the UCN bound states in the Earth's weak gravitational field can provide new insight regarding the verification of known interactions involving small distances.

An interesting view on the gravitation interaction, theoretically can lead us to the idea, elegantly declared in [21], regarding its universality and multi-dimension nature. In particular, it follows from [22] that a high dimensional correction using the framework of the Randall - Sundrum (RS) theory can be applied to Newton's gravitational law at small distances. To obtain a better understnding of this premise it is useful to evaluate such theoretical predictions from the experimental point of view. An interesting proposal related to the search of the RS correction to Newton's gravitational law on small scales was recently suggested by [23].

In this section our goal is to theoretically estimate the possible contribution the RS correction might have on Newton's gravitational law at small distances on the UCN weakly bound states in the Earth's gravitational field and to calculate an upper limit of the l_{RS} (Randall - Sundrum) parameter:

$$V(r) = \begin{cases} -\gamma Mm\left[\frac{1}{r}\left(1+\frac{l_{RS}^2}{r^2}\right)-\frac{1}{R_0}\left(1+\frac{l_{RS}^2}{R_0^2}\right)\right], & r > R_0 \\ \\ \infty, & 0 \le r \le R_0 \end{cases}$$

Using an approach simular to the one applied in the previous section we find that in a spherically symmetric case that the equation (3) can be rewritten by using the $\frac{x}{R_0} \ll 1$ approximation as:

$$\frac{d^2 f}{dx^2} + \frac{2m}{\hbar^2}\left[E - mg\left(1 + 3\cdot\frac{l_{RS}^2}{R_0^2}\right)x\right]f = 0 \tag{19}$$

Introducing in a similar way as before we inject a new additional constant: $\Lambda_{RS} = \frac{l_{RS}}{R_0}$, therefore the last equation (19) can be rewritten as:

$$\frac{d^2 f}{dx^2} + (\tilde{\alpha} - \tilde{\beta}x)f = 0 \tag{20}$$

with redefinition of the α and β values from equation (5) it follows:

$$\tilde{\alpha} \equiv \alpha = \frac{2mE}{\hbar^2}; \quad \tilde{\beta} = 2\left(\frac{m}{\hbar}\right)^2 g\left(1 + 3\Lambda_{RS}^2\right) \tag{21}$$

As it was shown before the solution of the equation (20) can be written in the form:

$$f(\tilde{z}) = \tilde{c}_N\sqrt{\tilde{z}}\left[J_{\frac{1}{3}}\left(\frac{2\tilde{z}^{\frac{3}{2}}}{3\tilde{\beta}}\right) + J_{-\frac{1}{3}}\left(\frac{2\tilde{z}^{\frac{3}{2}}}{3\tilde{\beta}}\right)\right] \tag{22}$$

where $\tilde{z} = \tilde{\alpha} - \tilde{\beta}x$. Using a simple connection between β, and $\tilde{\beta}$ values we arrive at:

$$\tilde{\beta} = \beta\biggl(1 + 3\Lambda_{RS}^2\biggr) \tag{23}$$

further we can decribe the contribution of the Λ_{RS} term regarding of weakly bound states of the UCN in the Earth's gravitational field using the boundary condition at $x = 0$:

$$J_{\frac{1}{3}}\left(\frac{2\tilde{\zeta}^{\frac{3}{2}}}{3\tilde{\beta}}\right) + J_{-\frac{1}{3}}\left(\frac{2\tilde{\zeta}^{\frac{3}{2}}}{3\tilde{\beta}}\right) = 0 \tag{24}$$

From the previous equation by substituting a new variable $\tilde{\zeta} = \left(\frac{2\tilde{\alpha}^{\frac{3}{2}}}{3\tilde{\beta}}\right)$ the final results can be presented as:

$$\tilde{E}_n = \left(\frac{9m}{8}(g\hbar\tilde{\zeta}_n)^2(1 + 3\Lambda_{RS}^2)^2\right)^{\frac{1}{3}} = c_0\biggl(1 + 3\Lambda_{RS}^2\biggr)^{\frac{2}{3}}\tilde{\zeta}_n^{\frac{2}{3}} \tag{25}$$

where $\tilde{\zeta}_n$ are a root of the equation (24). From this expression it follows that Λ_{RS} is equal to:

$$\Lambda_{RS} = \frac{1}{\sqrt{3\tilde{\zeta}_n}}\left(\left(\frac{\delta\tilde{E}_n}{c_0}\right)^{\frac{3}{2}} - 1)\right)^{\frac{1}{2}} \tag{26}$$

from which we can estimate, that

$$l_{RS} < 3.9 \cdot 10^3 m \tag{27}$$

V. QUANTUM MECHANICS APPLIED TO THE NS SURFACE.

Currently, one actual problem of great import in the area of astrophysics is the detection the Black Holes (BH) which could serve as an experimental proof of Einstein's gravitational theory. Another very important problem is the development of different precise methods for the measurement of the physical properties of distant astrophysical objects, such as: quasars, pulsars, gamma ray bursts, neutron stars (NS) and etc. In all these cases it would be very useful to provide such experimental investigations by different and completely independent physical methods. These experiments if successful would immediately raise the validity of the experimental data and provide clarification concerning the physical picture of the objects under investigation. Specifically, the greatest impact would occur in BH and NS physics. Within NS physics there may be a very interesting connection regarding the interplay of

general relativity (GR) and the quantum mechanical effects in the NS interior. The existence of the strange stars (SS) [24] underline that NS is a research area that is a very interesting and important for our further understanding of the behaviour of matter at extreme conditions. In this section we begin with a simple theoretical estimation because of the [11–15] approach, for the quantum effects on the NS surface. Introducing new notations: M_{NS} - NS mass, M_E - Earth mass, we can derive using the framework of the nonrelativistic approach of gravitational acceleration on the NS surface which is equal to:

$$g_{NS} = \left(\frac{M_{NS}}{M_E}\right)\left(\frac{R_E}{R_{NS}}\right)^2 \cdot g = \eta g \tag{28}$$

where R_E and R_{NS} are the Earth's and the NS radii respectively and η is defined as $\eta = \left(\frac{M_{NS}}{M_E}\right)\left(\frac{R_E}{R_{NS}}\right)^2$. If the density of neutron matter on the NS surface is $\rho = mn_\rho$, then we can simply estimate an average size of the cell, which would contain one neutron as:

$$a = \left(\frac{m}{\rho}\right)^{\frac{1}{3}} \geq \lambda_c = \frac{\hbar}{m_{\pi^0} c} \tag{29}$$

where m_{π^0} - is a π^0 meson mass, ρ - is the neutron matter density on the NS surface, n_ρ - its concentration, λ_c is the Compton wavelength and a - the size of the cell. In cases in which the cell size is larger than the π^0 meson Compton wavelength, in that it is approximately equivalent to the short range of nuclear forces, we can conclude that neutrons on the NS surface can be considered in the first step approximation as a free gas in the external gravitational field of the NS. This is because the average distance between them is larger than the radius of the nuclear interaction.

In our consideration, we suggest, that due to the Pauli exclusion principle there should exist, close to the NS surface, an analogy of the Fermi like surface, which can play the role of a mirror as is the case of the experiments conducted with the UCN in the Earth's gravitational field [11–15].

In this case, for rough estimation, we can simply follow the expressions derived before for the Earth's gravitational field with small modification to obtain the final result for the NS. The final expression, after such a modification of expressions (11), would appear as follows:

$$E_n^{(NS)} = \left(\frac{9m}{8}(\alpha_n g\hbar)^2\right)^{\frac{1}{3}} \cdot \eta^{\frac{2}{3}} \to E_n^{(Earth)}\eta^{\frac{2}{3}} \tag{30}$$

Finally, a simple example to illustrate the calculations described earlier related to the NS with given parameters is provided, the chosen parameters follow: $M_{NS} = 2 \cdot M_{Sun} \approx 4 \cdot 10^{30}$

TABLE II. Comparison of the energy of neutron's bound states (in eV units) on the NS and Earth's surface

n	E_n on NS	E_n, on Earth (th.)
1	$5.68 \cdot 10^{-5}$	$1.40 \cdot 10^{-12}$
2	$9.92 \cdot 10^{-5}$	$2.46 \cdot 10^{-12}$
3	$1.34 \cdot 10^{-4}$	$3.32 \cdot 10^{-12}$
4	$1.65 \cdot 10^{-4}$	$4.08 \cdot 10^{-12}$
5	$1.93 \cdot 10^{-4}$	$4.77 \cdot 10^{-12}$

kg, $R_{NS} \approx 10^4$ m and $M_E \approx 5.98 \cdot 10^{24}$ kg, $R_E \approx 6.37 \cdot 10^6$ m. The results of such calculations are presented in Table 2. For these numerical data $\eta \approx 2.7 \cdot 10^{-11}$. From the data in Table 2 we can conclude, that perhaps it is possible to discuss the possibility of NS spectroscopy, where the transition wave lengths will be around $\lambda_{tr} \sim \frac{2\pi\hbar c}{\Delta E_{tr}} \sim (0.124 - 1.24)$ cm, and the according transition energy $\Delta E_{tr} \sim (10^{-5} - 10^{-4})$ eV. Details of this spectroscopic information are directly connected to the NS physical parameters, such as the M_{NS} and R_{NS} values. The probability of their spectroscopic transitions $n \to k$ can be described as an overlap of the corresponding wave functions as described by the simple relation:

$$\omega_{n\to k} = \left(\int \psi_n(r)\psi_k(r) dr \right)^2 \tag{31}$$

VI. CONCLUSIONS

The present research, which used the works of [11–15] as a foundation devised a simple manner in which to consider the energy of UCN weakly bound quantum states in the Earth's gravitational field. Comparison of our results with the results of the above cited papers is presented in the Table 1.

The obtained results were generalized using the case of Newton's gravitation law correction by applying the RS theory [22] in the framework of a nonrelativistic approach. It was determined that in the further development of measurement techniques of the UCN in the Earth's gravitational field, we can more precisely estimate the upper limit of the contribution of the RS correction to Newton's gravitational law at a small distances. The contribution to the UCN energy spectrum from the RS correction to the formation of their bound states

was derived and the upper limit was estimated as $l_{RS} < 3.9 \cdot 10^3 m$. The ramifications of this are that the experimental methodology probably is not sensitive enough to check for a RS type correction to Newton's gravitational law at small distances.

The second step involved the estimation of neutrons bound state formation in the external gravitational field of a NS in comparison to the Earth's gravitational field neglecting GR, magnetic and rotation effects. In this case the results obtained are presented in Table 2, which presents some sample NS parameters and compares them with theoretical results obtained in the Earth gravitational field.

It is obvious, that when considering the neutron bound state formation on the NS surface it is necessary to apply a complete GR evaluation of the problem. The first and transparent approach in this direction is a reconsideration of the SE solution for the formation of a neutron bound state in the form:

$$g^{ij}\Big(\nabla_i\nabla_j - \Gamma^k_{ij}\nabla_k\Big)\psi + \frac{2m}{\hbar^2}\Big(E - U_{eff}(r)\Big)\psi = 0 \qquad (32)$$

where g^{ij} is a metric of the gravitational source (for example NS) and $U_{eff}(r)$ is its effective external gravitational field. Here i, j, k are spatial indexes. More mathematically rigorous considerations would need to use the Dirac wave equation in the gravitational field of the NS. Of course this approach would need to take into account magnetic fields and rotation effects, but is beyond the scope of this work in which the goal was to present some preliminary theoretical estimations. Application of such an approach can be very useful because of the potential of considering NS radial oscillation [25] which can lead to changes in the gravitational field on the NS surface. Also, there may be some observable effects related to neutron spectroscopy. In conclusion, we want to mention the paper [26], where the UCN storage problem was considered using the previously mentioned magnetic mirror in the presence of a gravitation field.

ACKNOWLEDGMENTS

A.V.K. wants to express his deep gratitude to Naresh Dadhich and Ajit Kembhavi from the Inter University Center for Astronomy and Astrophysics (IUCAA, Pune, India) for their invitation, warm hospitality, many useful discussions and clarifying the physical aspects of the problem. A.V.K. also wishes to thank the Bogoliubov Laboratory of Theoretical Physics

(Dubna, Russia) for the invitation and warm hospitality, where this work was finished. In addition A.V.K. expresses his thanks for many useful remarks (for example remarks about Fermi surface on the NS as a mirror for neutrons) at the Low Energy Nuclear Physics Seminar at the Bogoliubov Lab. of Theoretical Physics (JINR, Dubna, Russia) where this work was presented. R.A.S. and D.G. are grateful for a partial financial support for this work by the St. Cloud State University (St. Cloud, Minnesota, USA) internal grant program.

---

[1] Ya.B. Zeldovich, *JETP*, **36**, 1952, (1959).

[2] F.L. Shapiro, *Preprint JINR*, P3-7135, (1971).

[3] A.I. Frank, *Physics - Uspekhi*, **161**(11), 95, (1991).

[4] A.I. Frank, *Physics - Uspekhi*, **179**(4), 424, (2003).

[5] I.M. Frank, *Bulletin of USSR AS*, **41**(12), 13, (1971).

[6] V.I. Luschikov, Yu.N. Pokotilovsky, A.V. Strelkov and F.L. Shapiro, *JETP Lett.*, **9**, 23 (1969)

[7] F.L. Shapiro, *Physics - Uspekhi*, **95**, 145, (1968).

[8] V.I. Luschikov, A.I. Frank,*JETP Lett.*, **28**, 559 (1978)

[9] R. Colella, A.W. Overhauser, S.A. Werner, *Phys. Rev. Lett.*, **34**, 1472 (1975)

[10] H. Rauch, S.A. Werner, *Neutron Interferometry: Lessons in Experimental Quantum Mechanics*, (Clarendon Press, new York, 2000)

[11] V.V. Nesvizhevsky et al., *Nucl. Instrum. Method. A*, **440**(3), 754 (2000)

[12] V.V. Nesvizhevsky et al., *Nature*, **415**, 297 (2002)

[13] V.V. Nesvizhevsky et al., *Phys. Rev. D*, **67**, 102002 (2003)

[14] V.V. Nesvizhevsky et al., *Physics - Uspekhi*, **173**, 102 (2003)

[15] V.V. Nesvizhevsky et al., *Physics - Uspekhi*, **174**(5), 569 (2004)

[16] V.V. Nesvizhevsky and K.V. Protasov., in *Progress in Quantum Gravity*, (Frank Columbus, Nova, 2005)

[17] I.I. Goldman et al., *Problems in Quantum Mechanics*, (Ed. by D. Ter Haar, Academic Press, New York, 1960)

[18] J.J. Sakurai, *Modern Quantum Mechanics*, (Benjamin/Cummings, Menlo Park, California, 1985)

[19] P.C.W. Davies, *Quantum Mechanics*, (Guernsey Press Co.Ltd, London, 1984)

[20] G.N. Watson, *A Triatise on the theory of Bessel functions* , (Cambridge University Press, Cambridge, reprinted in 1996)

[21] Naresh Dadhich, gr-qc/0802.3034

[22] L. Randall and R. Sundrum,*Phys. Rev. Lett.*, **83**, 4690 (1999)

[23] M. Azam, M. Sami, C.J. Unnikrishnan, T. Shiromitzu, *Phys. Rev. D*, **77**, 101101 (2008)

[24] F. Weber, M. Meixner, R.P. Negreiros, astro-ph/0606093

[25] M. Gabler, U. Sperhake, N. Andersson, *Phys. Rev. D*, **80**, 064012 (2009)

[26] A.I. Frank, V.G. Nosov*JETP Lett.*, **79**(7), 387 (2004)